Around 3D-consistency

V.E. Adler (L.D. Landau ITP)

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Abstract

- The 3D-consistency property is usually formulated as the Consistency-Around-a-Cube for discrete equations on a square lattice (quad-equations). However, this property can be extended to some other types of equations, including continuous ones.

- In my talk, I will show that a multidimensional lattice governed by consistent quad-equations can carry some derivations that preserve this lattice and commute with each other. They are described by continuous equations of the KdV type and differential-difference equations of the Volterra lattice and dressing chain types, which are no less important objects than quad-equations.

- In principle, these equations can be obtained from quad-equations by continuous limits, but in my talk I will move in the opposite direction, interpreting quad-equations as the superposition formula for Bäcklund transformations.

- Particular attention will be paid to the interpretation of Volterra-type equations as negative symmetries for KdV-type equations and to the definition of 3D-consistency property for these symmetries.

Outline

$$u_t = u_{xxx} + f(u, u_x, u_{xx})$$ KdV type

$$u_{n,z} = g(u_{n-1}, u_n, u_{n+1})$$ Volterra lattice type

$$h(u_n, u_{n+1}, u_{n,x}, u_{n+1,x}; \alpha) = 0$$ dressing chain type

$$u_{xxz} = F(u, u_x, u_{xx}, u_z, u_{xz}; \alpha)$$ negative symmetry

$$Q(u_{n_i,n_j}, u_{n_i+1,n_j}, u_{n_i,n_j+1}, u_{n_i+1,n_j+1}; \alpha_i, \alpha_j) = 0$$ quad-equation

Main claim: there exist consistent sets of such equations. Presumably, such a set can be attached to any integrable KdV type equation.

Consistency means that it is possible to construct a function

$$u(x, t, z_1, z_2, \dots, n_1, n_2, \dots)$$

which satisfies all these equations with generic initial data given along x- and n_i-axes:

$$u(x,0,0,0) = \varphi(x), \quad u(0,0,0,0_{\hat{i}}) = \varphi_i(n_i).$$

Simplest example

$$u_t = u_{xxx} - 6u_x^2 \tag{1}$$

$$u_{n,z} = \frac{\beta}{u_{n+1} - u_{n-1}} \tag{2}$$

$$u_{n+1,x} + u_{n,x} = (u_{n+1} - u_n)^2 + \alpha \tag{3}$$

$$u_{xxz} = \frac{u_{xz}^2 - \beta^2}{2u_z} + 2(2u_x - \alpha)u_z \tag{4}$$

$$(u_{n_i,n_j} - u_{n_i+1,n_j+1})(u_{n_i+1,n_j} - u_{n_i,n_j+1}) = \alpha_i - \alpha_j \tag{5}$$

(1): the potential Korteweg–de Vries equation (pot-KdV)
(2): a particular case of $V_4(0)$ equation from the Yamilov list
(3): equivalent to the dressing chain for Schrödinger operator
(4): the associated Camassa–Holm equation
(5): H_1 from the Adler–Bobenko–Suris list

Remarks

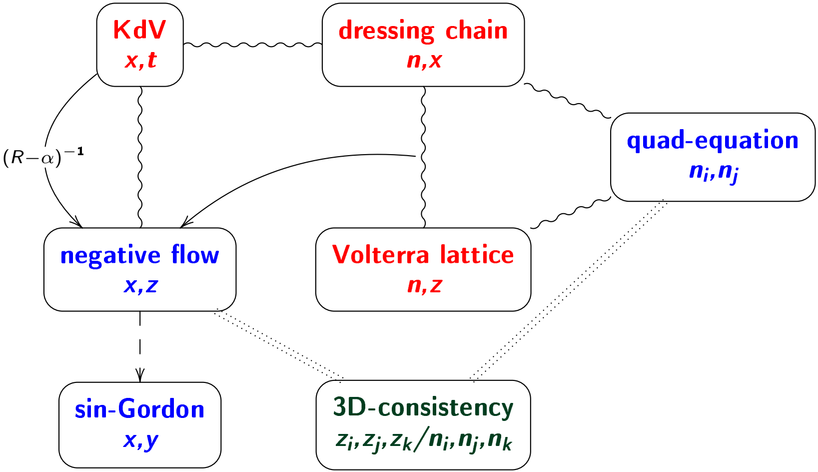
In this talk, we consider KdV, dressing chain and VL as basic building blocks from which quads and negs are derived:

- quad-equation is the superposition formula for Bäcklund transformations defined by dressing chain;
- negative flow is obtained from VL and dressing chain by elimination of n (associated system, in terminology by D. Levi).

However, this is not the only possible point of view. Vice versa, one can consider quad-equations and negative flow as main objects and starting points:

- all other equations can be obtained from the quad-equation by continuous limits;
- negative symmetry is a generating function for the whole KdV-type hierarchy.

Big picture



KdV
x,t
dressing chain
n,x
quad-equation
n_i,n_j
$(R-\alpha)^{-1}$
negative flow
x,z
Volterra lattice
n,z
sin-Gordon
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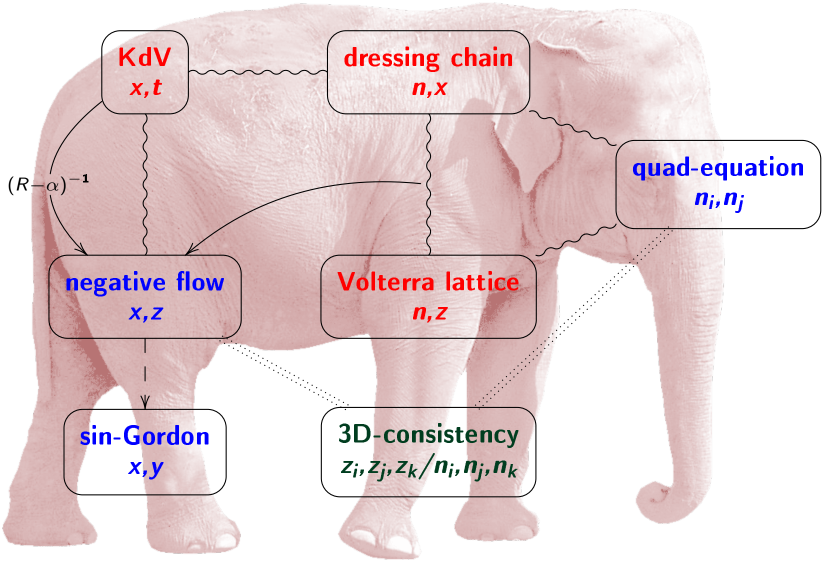
3D-consistency
$z_i,z_j,z_k/n_i,n_j,n_k$

Big picture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The full picture is even bigger and includes also

- noncommutative symmetries from additional subalgebra (master-symmetries);
- NLS-type systems associated with higher symmetries of VL (or symmetries of $u_{xxz} = \dots$ with respect to z-characteristic).

The talk is based mainly on the papers

[1] V.A., A.B. Shabat. Toward a theory of integrable hyperbolic equations of third order. *J. Phys. A: Math. Theor.* **45** (2012) 395207.

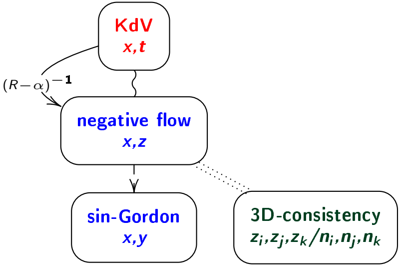
[2] V.A., M.P. Kolesnikov. Non-autonomous reductions of the KdV equation and multi-component analogs of the Painlevé equations P_{34} and P_3. *J. Math. Phys.* **64** (2023) 101505.

[3] V.A. Negative flows and non-autonomous reductions of the Volterra lattice. *Open Comm. in Nonl. Math. Phys., Special Issue in Memory of Decio Levi* (2024) 11597.

[4] V.A. Negative flows for several integrable models. *J. Math. Phys.* **65** (2024) 023502.

[5] V.A. 3D consistency of negative flows. *Theor. Math. Phys.* **221:2** (2024) 1836–1851.

Purely continuous point of view



KdV type equations (S-integrable)

$$u_t = u_{xxx} + uu_x \tag{KdV}$$

$$u_t = u_{xxx} + u^2 u_x \tag{mKdV}$$

$$u_t = u_{xxx} + u_x^2 \tag{pot-KdV}$$

$$u_t = u_{xxx} - \frac{1}{2}u_x^3 + (\alpha e^{2u} + \beta e^{-2u})u_x \tag{exp-CD}$$

$$u_t = u_{xxx} - \frac{3u_x(u_{xx} + r'(u))^2}{2(u_x^2 + 2r(u))} + r''(u)u_x, \quad r^{(5)} = 0 \tag{$\wp$-CD}$$

$$u_t = u_{xxx} - \frac{3(u_{xx}^2 + r(u))}{2u_x}, \quad r^{(5)} = 0 \tag{KN}$$

$$u_t = u_{xxx} - \frac{3u_x u_{xx}^2}{2(u_x^2 + 1)} + \alpha(u_x^2 + 1)^{3/2} + \beta u_x^3$$

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{4u_x} + \alpha u_x^{3/2} + \beta u_x^2$$

- Integrability is understood as existence of infinite hierarchy of higher symmetries.
- A complete list includes also several C-integrable (linearizable) equations.
- All equations from the list are related to KdV by differential substitutions, with the exception of KN.

[6] S.I. Svinolupov, V.V. Sokolov. Evolution equations with nontrivial conservation laws. *Funct. Anal. Appl.* **16:4** (1982) 317–319.

[7] A.V. Mikhailov, A.B. Shabat, V.V. Sokolov. The symmetry approach to classification of integrable equations. *in: V.E. Zakharov (ed). What is Integrability? Springer-Verlag, 1991*, pp. 115–184.

[8] A.G. Meshkov, V.V. Sokolov. Integrable evolution equations with constant separant. *Ufa Math. J.* **4:3** (2012) 104–154.

Hyperbolic symmetries

It is well known that *many* KdV type equations are consistent with equations of the form

$$u_{xy} = h(u, u_x, u_y). \tag{6}$$

The most famous example is, of course, sine-Gordon + pot-mKdV:

$$u_{xy} = \sin 2u, \qquad u_t = u_{xxx} + 2u_x^3, \qquad u_\tau = u_{yyy} + 2u_y^3.$$

Here, consistency = commutativity of the derivations D_x, D_y, D_t, D_τ acting on functions of dynamical variables $u, u_x, u_{xx}, \dots, u_y, u_{yy}, \dots$.

Paper [9] contains an exhaustive list of such triples.

However, there is no such symmetry for the KdV equation itself.

Why?

[9] A.G. Meshkov, V.V. Sokolov. Hyperbolic equations with third-order symmetries. *Theor. Math. Phys.* **166:1** (2011) 43–75.

Negative symmetries

A lesser known fact is that *any* (integrable) KdV type equation is consistent with an equation of the form

$$u_{xxz} = F(u, u_x, u_{xx}, u_z, u_{xz}; \alpha). \tag{7}$$

Again, this means that D_x, D_z, D_t commute on the extended dynamical set $u, u_x, u_{xx}, u_{xxx}, \dots, u_z, u_{xz}$.

In this case, we say that (7) defines a ***negative symmetry*** for the KdV type equation. This terminology is due to the fact that (7) can be derived with the help of negative power of recursion operator, as we will see soon.

It turns out that hyperbolic equations (6) are special reductions (which do not always exist) of (7).

Remarks

- For equations with variable separant (like Dym equation $u_t = u^3 u_{xxx}$), equation (7) may involve u_{xxx}.
- In this talk, we do not touch the symmetry of (7) with respect to the z-characteristic (which is equivalent to some NLS-type equation).

Example 1. The pot-mKdV $u_t = u_{xxx} + 2u_x^3$ is consistent with equation

$$u_{xxz} = 2u_x\sqrt{\alpha u_z^2 + 2\beta u_z + \gamma - u_{xz}^2} + \alpha u_z + \beta$$

for any α, β, γ. In the particular case $\alpha = \beta = 0$, it admits a first integral. We denote $z = y$ for this set of parameters and set $\gamma = 1$ by scaling. Then

$$\frac{u_{xxy}}{\sqrt{1 - u_{xy}^2}} = 2u_x \quad \Leftrightarrow \quad \arcsin u_{xy} = 2u + C(y)$$

and the change $u + C(y)/2 = \tilde{u}$ (admissible by pot-mKdV) brings to the sine-Gordon equation.

Example 2. The KdV equation $u_t = u_{xxx} - 6uu_x$ is consistent with

$$u_{xxz} = \frac{u_x}{2(u-\alpha)}\Big(u_{xz} + \sqrt{u_{xz}^2 - 4(u-\alpha)(u_z^2 - \gamma)}\Big) + 4(u-\alpha)u_z \qquad (8)$$

for any α, γ. However, there are no reductions in this case, for any values of parameters, which follows from the classification results of [9].

Negative symmetry $= (R-\mu)^{-1}(0)$

The origin of (7) is quite simple: given any integrable evolution equation

$$u_t = F$$

admitting a recursion operator R, one can define the flow

$$u_z = (R-\mu)^{-1}(0) \qquad \Leftrightarrow \qquad R(u_z) = \mu u_z \tag{9}$$

where μ is an arbitrary parameter. Since $u_{t_0} = 0$ is a trivial symmetry, (9) should be symmetry as well, by definition of R.

The only problem is that it is not local. However, we can somehow rewrite it, abandoning the evolutionary form of the equation.

❦ This works also for NLS, Boussinesq and other equations [10, 11, 4].

[10] A.M. Kamchatnov, M.V. Pavlov. On generating functions in the AKNS hierarchy. *Phys. Lett. A* **301:3–4** (2002) 269–274.

[11] S.Y. Lou, M. Jia. From one to infinity: symmetries of integrable systems. *JHEP* **02** (2024) 172.

Simplest example: pot-KdV

Let us consider again the KdV equation:

$$u_t = u_{xxx} - 6uu_x. \tag{KdV}$$

The recursion operator

$$R = D_x^2 - 4u - 2u_x D_x^{-1}$$

generates the positive part of the hierarchy:

$u_{t_1} = R(0) = u_x$	translation on x
$u_{t_2} = R(u_x) = u_{xxx} - 6uu_x$	KdV
$u_{t_3} = R^2(u_x) = (u_{xxxx} - 10uu_{xx} - 5u_x^2 + 10u^3)_x, \quad \dots$	higher symmetries

The negative symmetry (with $\mu = -4\alpha$) reads $(R + 4\alpha)(u_z) = 0$, that is

$$u_{xxz} - 4(u - \alpha)u_z - 2u_x D_x^{-1}(u_z) = 0 \quad \Leftrightarrow \quad u_{xxxz} = \dots \tag{10}$$

This can be once integrated and (8) appears with γ as the integration constant.

Equation (8) is somewhat disappointing: it is too cumbersome compared to the KdV equation itself. We can cast it into much simpler form by passing to the potential v by substitution

$$u = 2v_x, \qquad q = D_x^{-1}(u_z) = 2v_z.$$

Then (10) takes the form

$$v_{xxxz} - 4(2v_x - \alpha)v_{xz} - 4v_{xz}v_z = 0$$

and integrating with factor v_z brings to the following consistent pair:

$$v_t = v_{xxx} - 6v_x^2, \qquad \text{(pot-KdV)}$$

$$2v_z v_{xxz} - v_{xz}^2 - 4(2v_x - \alpha)v_z^2 + \gamma = 0. \qquad (11)$$

The latter equation is known as the associated Camassa–Holm equation.

[12] J. Schiff. The Camassa–Holm equation: a loop group approach. *Physica D* **121:1–2** (1998) 24–43.

[13] A.N.W. Hone. The associated Camassa–Holm equation and the KdV equation. *J. Phys. A: Math. Gen.* **32:27** (1999) L307–314.

Generating function for higher symmetries

An important property of a negative symmetry is that its formal expansion in powers of μ can be viewed as a generating function for higher symmetries:

$$u_z = (R-\mu)^{-1}(0) = -\mu^{-1}(1+\mu^{-1}R+\mu^{-2}R^2+\dots)(0)$$
$$= -\mu^{-2}u_{t_0} - \mu^{-3}u_{t_1} - \mu^{-4}u_{t_2} - \dots$$

In particular, the higher symmetries of KdV are of the form $u_{t_i} = q_x^{(i+1)}$ where $q^{(i)}$ are coefficients of the formal series

$$q = q^{(0)} + \frac{q^{(1)}}{\mu} + \frac{q^{(2)}}{\mu^2} + \dots, \quad q^{(0)} = -\frac{1}{2}$$

which satisfies (11) with $2v_x = u$ and $2v_z = q$:

$$2qq_{xx} - q_x^2 - (4u+\mu)q^2 + \frac{\mu}{4} = 0.$$

This is equivalent to the well-known quadratic recurrence relations

$$q^{(i+1)} = \sum_{s=0}^{i}\bigl(q_x^{(s)}q_x^{(i-s)} - 2q^{(s)}q_{xx}^{(i-s)} + 4uq^{(s)}q^{(i-s)}\bigr) + \sum_{s=1}^{i} q^{(s)}q^{(i+1-s)}.$$

- In fact, q is the resolvent of the Sturm–Liouville operator $L=-D_x^2+u$, with μ playing the role of spectral parameter [14].
- Also, q can be represented as $q=\psi\varphi$ where $L\psi=\mu\psi$, $L\varphi=\mu\varphi$ (squared eigenfunction symmetry) [15].

[14] I.M. Gel'fand, L.A. Dikii. Asymptotic properties of the resolvent of Sturm–Liouville equations, and the algebra of Korteweg–de Vries equations. *Russian Math. Surveys* **30:5** (1975) 77–113.

[15] A.Yu. Orlov, S. Rauch-Wojciechowski. Dressing method, Darboux transformation and generalized restricted flows for the KdV hierarchy. *Physica D* **69:1–2** (1993) 77–84.

Do negative symmetries commute with each other?

We know that positive higher symmetries of KdV are commutative.

The negative flow for KdV contains the parameter α and the flows with different α are different (other parameters are not very important).

It is clear that these flows *must* commute, because they can be viewed as series in α with coefficients given by commutative higher symmetries.

Conversely, if negative symmetries with different α are commutative, then this implies commutativity of the whole positive hierarchy.

Is it possible to prove the commutativity of negative symmetries independently, without resorting to recursion operators and higher symmetries?

Let us forget about KdV-type equations at all. Let equations be given

$$u_{xxz_i} = F_i(u, u_x, u_{xx}, u_{xxx}, u_{z_i}, u_{xz_i}), \quad i \in I;$$

we wish to define a correct notion of consistency for such a set of equations.

Example: Ferpontov's triple of hyperbolic equations

The following triple is 3D consistent:

$$u_{xy}=\sinh u\sqrt{1+u_x^2},\quad u_{yz}=\cosh u\sqrt{1+u_z^2},\quad u_{xz}=\sqrt{1+u_x^2}\sqrt{1+u_z^2}.$$

The cross derivatives for each pair of equations coincide, assuming that the rest equation holds. For instance, for the first and second eqs:

$$(u_{xy})_z-(u_{xz})_y=(u_{xz}-\sqrt{1+u_x^2}\sqrt{1+u_z^2})\left(\frac{u_z\cosh u}{\sqrt{1+u_z^2}}-\frac{u_x\sinh u}{\sqrt{1+u_x^2}}\right),$$

which vanishes due to the third equation. The third equation is recovered in this way, since the second factor contains lower derivatives and can be cancelled. The same is true for any pair.

In other words, in order to define consistency of two equations, we *have* to include the third equation into the definition. The same idea works for u_{xxz}-equations, with technical complications.

[16] E.V. Ferapontov. Laplace transformations of hydrodynamic-type systems in Riemann invariants: periodic sequences. *J. Phys. A* **30:19** (1997) 6861–6878.

3D consistency of negative symmetries

Definition. Equations

$$u_{xxz_i} = F_i(u, u_x, u_{xx}, u_{xxx}, u_{z_i}, u_{xz_i}), \quad i \in I, \tag{12}$$

are 3D consistent if there exist additional equations

$$u_{z_i z_j} = G_{ij}(u, u_x, u_{xx}, u_{z_i}, u_{xz_i}, u_{z_j}, u_{xz_j}), \quad i \neq j, \tag{13}$$

such that $G_{ij} = G_{ji}$ and for different $i, j, k \in I$,

$$D_{z_i}(F_j) = D_{z_j}(F_i) = D_x^2(G_{ij}), \tag{14}$$

$$D_{z_i}(G_{jk}) = D_{z_j}(G_{ik}) = D_{z_k}(G_{ij}), \tag{15}$$

identically in virtue of (12), (13) and the differential consequences $u_{xxxz_i} = D_x(F_i)$, $u_{xz_i z_j} = D_x(G_{ij})$.

This implies the coincidence of *any* mixed derivatives, which guarantees the existence of local simultaneous equations for the whole set of equations.

Is this definitions constructive? Yes: equations (13) are not given in advance, but they can be recovered, if they exist.

First step: examine the condition $(u_{xxz_i})_{z_j} = (u_{xxz_j})_{z_i}$, that is

$$0 = D_{z_i}(F_j) - D_{z_j}(F_i) = P_{ij}(u, u_x, u_{xx}, u_{xxx}, u_{z_i}, u_{xz_i}, u_{z_j}, u_{xz_j}, u_{z_i z_j}, u_{xz_i z_j})$$

where the derivatives u_{xxxz} and u_{xxz} in the r.h.s. are eliminated by (12). By solving this with respect to $u_{xz_i z_j}$, we get

$$u_{xz_i z_j} = H_{ij}(u, u_x, u_{xx}, u_{xxx}, u_{z_i}, u_{xz_i}, u_{z_j}, u_{xz_j}, u_{z_i z_j}). \qquad (16)$$

This should be a corollary of (13) and this gives the next condition.

Second step: examine the condition $(u_{xz_i z_j})_x = (u_{xxz_i})_{z_j}$, that is

$$0 = D_x(H_{ij}) - D_{z_j}(F_i) = Q_{ij}(u, u_x, u_{xx}, u_{xxx}, u_{xxxx}, u_{z_i}, u_{xz_i}, u_{z_j}, u_{xz_j}, u_{z_i z_j})$$

where u_{xxxz}, u_{xxz} and $u_{xz_i z_j}$ are eliminated. By solving with respect to $u_{z_i z_j}$, we obtain the desired equation (13).

Final step: check the equalities $(u_{z_i z_j})_x = u_{xz_i z_j}$ and $(u_{z_i z_j})_{z_k} = (u_{z_i z_k})_{z_j}$ by straightforward calculation.

Example: pot-KdV

Proposition. The following equations are 3D consistent:

$$u_{xxz_i} = \frac{u_{xz_i}^2 - \gamma_i}{2u_{z_i}} + 2(2u_x - \alpha_i)u_{z_i}, \tag{17}$$

$$u_{z_iz_j} = \frac{u_{z_i}u_{xz_j} - u_{z_j}u_{xz_i}}{\alpha_i - \alpha_j}, \quad \alpha_i \neq \alpha_j. \tag{18}$$

Let us illustrate the algorithm of derivation of additional eqs (18). On the first step, $(u_{xxz_i})_{z_j} = (u_{xxz_j})_{z_i}$ provides an expression for $u_{xz_iz_j}$:

$$\begin{aligned} u_{xz_iz_j} &= \left(2(\alpha_i - \alpha_j)u_{z_i}u_{z_j} + \frac{\gamma_j u_{z_i}}{2u_{z_j}} - \frac{\gamma_i u_{z_j}}{2u_{z_i}}\right)\frac{u_{z_iz_j}}{u_{z_j}u_{xz_i} - u_{z_i}u_{xz_j}} \\ &\quad + \frac{1}{2}\left(\frac{u_{xz_i}}{u_{z_i}} + \frac{u_{xz_j}}{u_{z_j}}\right)u_{z_iz_j} + 4u_{z_i}u_{z_j}. \end{aligned} \tag{19}$$

Second step: $(u_{xz_iz_j})_x = (u_{xxz_i})_{z_j}$ yields the *factorized* equation (cf. with the Ferapontov's example)

$$((\alpha_i-\alpha_j)u_{z_iz_j}+u_{z_j}u_{xz_i}-u_{z_i}u_{xz_j})\times \\ \times \frac{(u^2_{z_i}u^2_{xz_j}-u^2_{z_j}u^2_{xz_i}-4(\alpha_i-\alpha_j)u^2_{z_i}u^2_{z_j}+\gamma_iu^2_{z_j}-\gamma_ju^2_{z_i})}{(u_{z_j}u_{xz_i}-u_{z_i}u_{xz_j})^2}=0.$$

By setting the first factor to zero, we obtain (18).

Next, we verify that $(u_{z_iz_j})_x = u_{xz_iz_j}$ is an identity. Derivation of (18) with respect to x gives

$$u_{xz_iz_j}=2u_{z_i}u_{z_j}+\frac{1}{2(\alpha_i-\alpha_j)}\left(\frac{u_{z_i}}{u_{z_j}}(u^2_{xz_j}-\gamma_j)-\frac{u_{z_j}}{u_{z_i}}(u^2_{xz_i}-\gamma_i)\right).$$

This coincides with (19) after substituting $u_{z_iz_j}$ from (18), that is (19) follows from (18) and (17).

Finally, we verify the identities (15), that is $(u_{z_iz_j})_{z_k}=(u_{z_iz_k})_{z_j}$ which completes the proof of 3D consistency.

Remark. Equation (18)

$$u_{z_i z_j} = \frac{u_{z_i} u_{xz_j} - u_{z_j} u_{xz_i}}{\alpha_i - \alpha_j}, \quad \alpha_i \neq \alpha_j$$

is an independent 3D integrable equation related with the Alonso–Shabat universal hydrodynamic hierarchy [17]. It satisfies the identity [18]

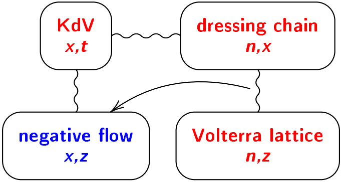
$$(u_{z_i z_j})_{z_k} = (u_{z_i z_k})_{z_j}.$$

To prove it, equations (17) are not needed: they only define a 2D reduction of this 3D equation.

[17] L. Martínez Alonso, A.B. Shabat. Hydrodynamic reductions and solutions of the universal hierarchy. *Theor. Math. Phys.* **140:2** (2004) 1073–1085.

[18] V.E. Adler, A.B. Shabat. Model equation of the theory of solitons. *Theor. Math. Phys.* ***153:1*** (2007) 1373–1387.

$$\text{Negative symmetry}(x, z; \alpha) = \frac{\text{Volterra lattice}(\not{n}, z)}{\text{dressing chain}(\not{n}, x; \alpha)}$$



Volterra type lattice equations

Main part of the list:

$$u_{n,z} = P(u_n)(u_{n+1} - u_{n-1}) \qquad V_1$$

$$u_{n,z} = P(u_n^2)\left(\frac{1}{u_{n+1}+u_n} - \frac{1}{u_n+u_{n-1}}\right) \qquad V_2$$

$$u_{n,z} = Q(u_n)\left(\frac{1}{u_{n+1}-u_n} + \frac{1}{u_n-u_{n-1}}\right) \qquad V_3$$

$$u_{n,z} = \frac{R(u_{n-1},u_n,u_{n-1}) + \nu R(u_{n+1},u_n,u_{n+1})^{\frac12} R(u_{n-1},u_n,u_{n-1})^{\frac12}}{u_{n+1}-u_{n-1}} \qquad V_4(\nu)$$

plus several more potential forms.

Here $\nu = 0, \pm 1$ and

$$P'''(u) = 0, \qquad Q^{(5)}(u) = \alpha u^4 + \beta u^3 + \gamma u^2 + \delta u + \varepsilon,$$

$$R(u,v,w) = (\alpha v^2 + 2\beta v + \gamma)uw + (\beta v^2 + \lambda v + \delta)(u+w) + \gamma v^2 + 2\delta v + \varepsilon.$$

- This list was obtained by Yamilov [19, 20] almost simultaneously with the classification of KdV type equations.
- There is a lot of parallels between the continuous and discrete theories, but there is also a lot of differences. In general, these classes of equations can be studied independently.
- Our goal is to demonstrate how one can put together the KdV and Volterra classes. To do this we need one more class of equations: dressing chains.

[19] R.I. Yamilov. On classification of discrete evolution equations. *Uspekhi Math. Nauk* **38:6** (1983) 155–156.

[20] R.I. Yamilov. Symmetries as integrability criteria for differential difference equations. *J. Phys. A* **39:45** (2006) R541–623.

Dressing chains

These are equations of the form

$$h(u_n, u_{n+1}, u_{n,x}, u_{n+1,x}; \alpha) = 0 \tag{20}$$

which can be viewed as differential-difference analog of hyperbolic equations $u_{xy} = h(u, u_x, u_y)$.

We assume that this equation can be solved with respect to the derivatives, so that it can be also written in two equivalent forms

$$u_{n+1,x} = a(u_{n+1}, u_n, u_{n,x}; \alpha), \quad u_{n-1,x} = b(u_{n-1}, u_n, u_{n,x}; \alpha).$$

Equation (20) may admit evolution symmetries of both KdV and Volterra type

$$u_t = u_{xxx} + f(u, u_x, u_{xx}), \qquad u_{n,z} = g(u_{n-1}, u_n, u_{n+1}).$$

This means that D_t or D_z derivative of (20) vanishes in virtue of this equation itself.

- The role of dressing chains as x-part of Bäcklund transformations for the KdV type equations is commonly known.
- The consistency of dressing chains with Volterra type equations was studied, e.g. in [21, 22, 23], but this topic is much less popular so far.

[21] R.I. Yamilov. Invertible changes of variables generated by Bäcklund transformations. *Theor. Math. Phys.* **85:3** (1990) 1269–1275.

[22] R.N. Garifullin, I.T. Habibullin, R.I. Yamilov. Peculiar symmetry structure of some known discrete nonautonomous equations. *J. Phys. A: Math. Theor.* **48:23** (2015) 235201.

[23] R.N. Garifullin, I.T. Habibullin. Generalized symmetries and integrability conditions for hyperbolic type semi-discrete equations. *J. Phys. A: Math. Theor.* **54:20** (2021) 205201.

Derivation of negative symmetry from lattice equations

Let a pair of consistent equations be given

$$h(u_n, u_{n+1}, u_{n,x}, u_{n+1,x}; \alpha) = 0, \qquad u_{n,z} = g(u_{n-1}, u_n, u_{n+1}). \tag{21}$$

We want to get rid of variables $u_{n\pm1}$ and demonstrate that $u = u_n$ satisfies an equation of the form

$$u_{xxz} = F(u, u_x, u_{xx}, u_z, u_{xz}; \alpha). \tag{22}$$

To do this, we first solve the first equation (21) and its copy for $n = n - 1$ with respect to $u_{n\pm1,x}$. Let

$$u_{n+1,x} = a(u_{n+1}, u_n, u_{n,x}; \alpha), \quad u_{n-1,x} = b(u_{n-1}, u_n, u_{n,x}; \alpha). \tag{23}$$

Then we differentiate second equation (21) with respect to x and replace $u_{n\pm1,x}$. This gives us two equations free of $u_{n\pm1,x}$:

$$u_{n,z} = g(u_{n-1}, u_n, u_{n+1}), \qquad u_{n,xz} = G(x, u_{n,x}, u_{n-1}, u_n, u_{n+1}),$$

which is a system with respect to u_{n-1} and u_{n+1}.

If the Jacobian of this system with respect to $u_{n\pm1}$ vanishes then both variables eliminate and we arrive to an equation for u_n of the form $u_{xz} = \dots$, which is a degenerate case of negative symmetry.

However, generically, the Jacobian is not zero and then it is possible to find u_{n+1} as a function of $u_n, u_{n,x}, u_{n,z}, u_{n,xz}$. Then differentiating once more and substituting to the first equation (23) yields (22).

In other words, the negative flow (22) can be viewed as a *quotient equation*: it is a Volterra type flow modulo dressing chain

$$\text{negative flow}(x, z; \alpha) = \frac{\text{Volterra lattice}(n, z)}{\text{dressing chain}(n, x; \alpha)}.$$

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Example: pot-KdV

The Bäcklund transformation for pot-KdV

$$u_t = u_{xxx} - 6u_x^2$$

is very well known [24]: it is given by the dressing chain

$$u_{n+1,x} + u_{n,x} = (u_{n+1} - u_n)^2 + \alpha.$$

One can easily check that these equations are consistent with each other and with $V_4(0)$ where $R(u,v,w) = \beta$

$$u_{n,z} = \frac{\beta}{u_{n+1} - u_{n-1}},$$

and that elimination of n brings to already familiar negative symmetry

$$u_{xxz} = \frac{u_{xz}^2 - \beta^2}{2u_z} + 2(2u_x - \alpha)u_z.$$

[24] H.D. Wahlquist, F.B. Estabrook. Bäcklund transformations for solutions of the Korteweg–de Vries equation. *Phys. Rev. Let.* **31:23** (1973) 1386–1390.

It turns out that the same Volterra lattice is also consistent with other equations: the Schwarzian-KdV

$$u_t = u_{xxx} - \frac{3u_{xx}^2}{2u_x},$$

and

$$u_{n,x}u_{n+1,x} = 1.$$

Of course, the latter one can hardly be considered as genuine Bäcklund transformation because it is integrated to the trivial change $u_{n+2} = u_n + \mathrm{const}$. However, formally these equations constitute a consistent triple and elimination of n brings in this case to equation

$$u_{xz} = 0$$

which is consistent with Schwarzian-KdV indeed.

The generic lattices consistent with Schwarzian-KdV are

$$u_{n,x}u_{n+1,x} = \alpha(u_n - u_{n+1})^2, \qquad u_{n,z} = \beta\frac{(u_{n+1} - u_n)(u_n - u_{n-1})}{u_{n+1} - u_{n-1}}$$

and the corresponding negative flow is

$$u_{xxz} = \frac{u_{xz}^2 - \beta^2 u_x^2}{2u_z} + \frac{u_{xx}u_{xz}}{u_x} + 2\alpha u_z.$$

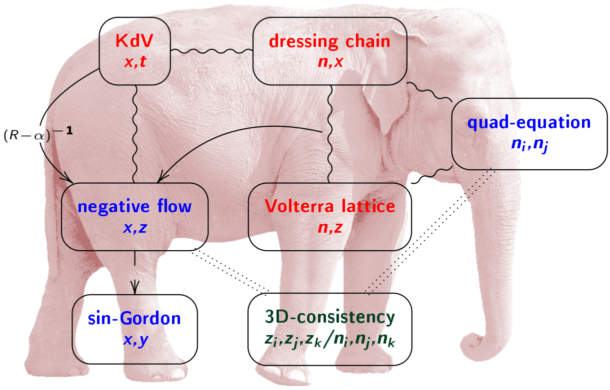
Conjecture

- Any equation from the KdV list is consistent with some dressing chain and a Volterra type lattice equation.
- Vice versa, any equation from the Volterra list is consistent with some dressing chain and a KdV type equation.

The open problem is to classify all possible consistent sets of these equations.

The previous example demonstrates that this correspondence between the lists is not a bijection. However, the deviations from bijection are probably related with 'degenerated' equations only.

Assembling all together



Explaining 3D-consistency

The appearance of dressing chains makes it clear that the 3D-consistency of negative symmetries is closely related to the 3D-consistency of quad-equations which define the nonlinear superposition principle for Bäcklund transformations.

For the pot-KdV example, the superposition formula for the dressing chain is

$$(u - T_iT_j(u))(T_i(u) - T_j(u)) = \alpha_i - \alpha_j \qquad (H_1)$$

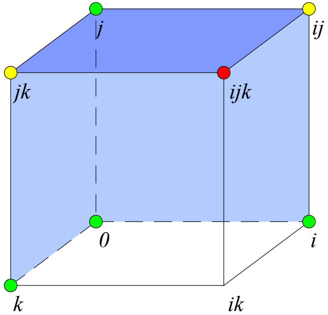
where $T_i : u(\dots, n_i, \dots) \mapsto u(\dots, n_i + 1, \dots)$.

These equations are defined for each pair i, j and should be consistent around a cube in order to define a generic solution on the multi-dimensional lattice $(n_1, n_2, \dots)$. Each variable n_i corresponds to the pair of equations

$$T_i(u_x) + u_x = (T_i(u) - u)^2 + \alpha_i, \quad u_{z_i} = \frac{\beta_i}{T_i(u) - T_i^{-1}(u)},$$

so that each coordinate n_i is associated with the negative flow ∂_{z_i} with parameters α_i, β_i. The consistency of negative flows becomes just a corollary of the consistency of these building blocks.

Consistency around a cube



Let us recall that for the discrete hyperbolic equations

$$u_{ij} = h(u, u_i, u_j; \alpha_i, \alpha_j)$$

the consistency is defined for a triple of equations defined on the faces of a cube. For short, we use subscript i to denote T_i:

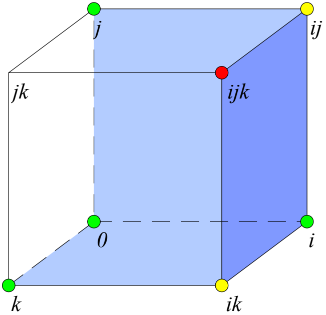
$$u_i = T_i(u) = u(\dots, n_i + 1, \dots).$$

Three possible ways to compute u_{ijk} from initial data u, u_i, u_j, u_k should give the same results.

A list of 3D-consistent equations was ontained in [25], under many additional assumptions.

[25] V.A., A.I. Bobenko, Yu.B. Suris. Classification of integrable equations on quad-graphs. The consistency approach. *Comm. Math. Phys.* ***233:3*** (2003) 513–543.

Consistency around a cube



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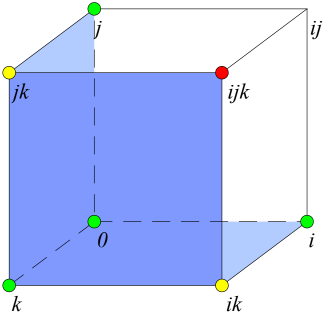

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List of 3D-consistent multi-affine quad-equations

$$(u - u_{ij})(u_i - u_j) + \alpha_j - \alpha_i = 0 \qquad (H_1)$$

$$\dots \quad H_2, H_3, A_1, A_2, Q_1, Q_2, Q_3 \quad \dots$$

$$\begin{aligned}\mathrm{sn}(\alpha_i;k)\,\mathrm{sn}(\alpha_j;k)\,\mathrm{sn}(\alpha_i-\alpha_j;k)(k^2uu_iu_ju_{ij}+1)+\mathrm{sn}(\alpha_i;k)(uu_i+u_ju_{ij})\\ -\mathrm{sn}(\alpha_j;k)(uu_j+u_iu_{ij})-\mathrm{sn}(\alpha_i-\alpha_j;k)(uu_{ij}+u_iu_j)=0\end{aligned} \qquad (Q_4)$$

- All these equations serve as nonlinear superposition formula for Bäcklund transformations of KdV type equations.
- Converse is not true, because the nonlinear superposition is not necessarily multi-affine (the simplest counter-example is just the KdV equation itself). Quad-equations which are quadratic with respect to all variables where studied, e.g. in [28, 29].

[28] P. Kassotakis, M. Nieszporski. On non-multiaffine consistent-around-the-cube lattice equations. *Phys. Lett. A* **376:45** (2012) 3135–3140.

[29] J. Atkinson, M. Nieszporski. Multi-quadratic quad equations: integrable cases from a factorised-discriminant hypothesis. *Int. Math. Res. Notices* **2014:15** (2013) 4215–4240.

- The consistency of Volterra type equations with quad-equations has been studied in many papers.

- The large open project is to classify the correspondence

$$\begin{array}{ccccc} \text{KdV type} & \longleftrightarrow & \text{dressing chains} & \longleftrightarrow & \text{Volterra type} \\ & & \updownarrow & & \\ & & \text{quad-equations} & & \end{array}$$

and to describe the negative flows in this language.

[30] F.W. Nijhoff, V.G. Papageorgiou. Similarity reductions of integrable lattices and discrete analogues of Painlevé PII equation. *Phys. Lett. A* **153:6–7** (1991) 337–344.

[31] F.W. Nijhoff, A. Ramani, B. Grammaticos, Y. Ohta. On discrete Painlevé equations associated with the lattice KdV systems and the Painlevé VI equation. *Studies in Appl. Math.* **106:3** (2001) 261–314.

[32] A. Tongas, D. Tsoubelis, P. Xenitidis. Integrability aspects of a Schwarzian PDE. *Phys. Lett. A* **284:6** (2001) 266–274.

[33] P.D. Xenitidis. Determining the symmetries of difference equations. *Proc. R. Soc. A* **474** (2018) 20180340.